American Option

Math 622

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1 Introduction

An American option purchased at time $t = 0$ with pay off function $g(x)$ and expiry T on an underlying stock $S(t)$ is a contract that gives the buyer the right, but not the obligation, to exercise the option at any time $t \in [0, T]$ and receive the pay off $g(S_t)$. If the option holder does not exercise at or before the expiry T then the option becomes worthless. We call $g(S_t)$ the *intrinsic value* of the option.

Example:
(i) American call option: $g(x) := (x - K)^+$;
(ii) American put option: $g(x) := (K - x)^+$;
(iii) Perpetual American put option: $g(x) := (K - x)^+$ and $T = \infty$.

Remark 1.1. *It is clear that if $g(S_t) < 0$ then the American option holder will not exercise the option at time t. Thus we can assume that $g(x) \geq 0$ for all x. Then $g(S_T) \geq 0$ and thus we can assume that the option is always exercised at or before time T.*

Remark 1.2. *We have the following important observation: Let V_t^A be the (no arbitrage) price of an American option and V_t^C the (no arbitrage) price of the corresponding European option purchased at time $t = 0$ with pay off function $g(x)$ and expiry T. That is $V_T^C = g(S_T)$ and the European option holder cannot exercise the option earlier than T. Then $V_t^A \geq V_t^C$. That is the price of an American option is always at least as expensive as its European counterpart. Note that this conclusion is model independent: we do not make any assumption on S_t.*

Reason: Suppose $V_t^C > V_t^A$. Then we take a long position (that is we buy 1 share) on the American option and a short position (that is we sell 1 share) on the European option. Then we make a risk free positive profit equals $V_t^C - V_t^A$. At time T, we exercise the American option to close out our short position on the European option. Thus this is an arbitrage opportunity and therefore we must have $V_t^C \leq V_t^A$.

Remark 1.3. *Another important observation is that when $g(x) = (x-k)^+$, then $V_t^A = V_t^C$ for all t and thus the optimal exercise time for the American option is at the expiry T if the asset S_t* **does not pay dividend**. *This obseration is also model-independent and the explaination is provided in the solution to homework 6. This conclusion does not extend to the case when S_t pays dividend.*

We now need a mathematical definition of the American option price.

2 Preliminary investigation

2.1 Exercise times

2.1.1 Exercise times on the time interval $[0, T]$

The first step in modeling American option is to formulate the notion of exercise time mathematically. Here we suppose that our current time is 0 and we are considering all the future possible exercise times for the option holder.
As usual, we assume the information available to investors as time progresses is encoded in a given filtration $\{\mathcal{F}(t);\, t \geq 0\}$. In general, the owner of an American option will decide when to exercise based on the current level of the price, its past history, and whatever other information is available about the economy and its past history. It is assumed that investors cannot look into the future. So at the moment an investor decides to exercise, he or she can use only the information in the filtration up to that moment. Mathematically, this translates into the following assumption:

$$\textit{all exercise times are } \{\mathcal{F}(t);\, t \geq 0\}\textit{-stopping times.} \tag{1}$$

Most of the analysis of American options that is presented in this course will be made under the following additional assumptions:

$$\text{the price process } S \text{ is a Markov process;} \tag{2}$$

$$\{\mathcal{F}(t);\, t \geq 0\} \text{ is the filtration generated by } S \tag{3}$$

The analysis we will do shows that, assuming (2) and (3), the best decision about whether to exercise or not at time t will use only the current value $S(t)$ of the price of the underlying. This is ultimately a consequence of the Markov property of S.

2.1.2 Exercise time on the time interval $[t, T]$

Consider now $t < T$. Here we want to look at all the possible exercise time beyond t for the option holder, *assuming the option has not been exercised before time t.* Since S is Markov, the value of $S(t)$ contains all the information from the past relevant to the future when computing conditional expectations. Define

$$\mathcal{F}^{(t)}(u) \quad \text{be the } \sigma\text{-algebra generated by } S(v) \text{ for times } t \leq v \leq u.$$

This is the filtration of information generated by S for times after t. It will thus be assumed of any exercise time taking place at time t or later that

$$t \leq \tau \leq T; \text{ and,} \tag{4}$$

$$\tau \text{ is an } \{\mathcal{F}^{(t)}(u);\, u \geq t\}\text{-stopping time.} \tag{5}$$

The meaning of condition (5) is that for every $u \geq t$, the event $\{t \leq \tau \leq u\}$ must belong to $\mathcal{F}^{(t)}(u)$.

2.2 The mathematical definition of the value of an American option

Let $\{(\Omega, \mathcal{F}, P), \{S(t);\, t \geq 0\}$ be a risk-neutral price model. Let $\{\mathcal{F}(t);\, t \geq 0\}$ be the filtration generated by S and assume the risk free interest rate is the constant r. Consider an American option that pays $g(S(t))$ if exercised at t and let T be its expiration date.
The following observations are important for us:

(i) $V(t) \geq g(S(t))$.

Reason: If $V(t) < g(S(t))$, one could buy the option and immediately exercise to realize a riskless positive profit $g(S(t)) - V(t)$.

(ii)Let τ be a given stopping time satisfying $P(\tau \leq T) = 1$. For example,

$$\tau = \inf\{t \geq 0 : S_t = L\} \wedge T,$$

where L is a positive constant. Consider a financial product that pays $g(S(\tau))$ at time τ (Note that this is not an American option - the payoff time (albeit being random) is specified in advance at time 0). According to risk-neutral pricing, the price V_0^τ of this product is

$$V_0^\tau = E\Big[e^{-r\tau} g(S(\tau))\Big].$$

Now consider an American option with payoff function $g(x)$ and expiry T. Let V_t denote the value of the option at time t. Then $V_0 \geq V_0^\tau$.

Reason: If $V_0 < V_0^\tau$, one could buy 1 share the American option and sell 1 share of the financial product. This results in a riskless positive profit of $V_0^\tau - V_0$ at time 0. At time τ, we simply excercise the American option to cover the payoff $g(S_\tau)$ from the financial product. This is an arbitrage opportunity.

(iv) It follows that V_0, the price of the American option, satisfies

$$V_0 \geq \sup\left\{E\Big[e^{-r\tau} g(S(\tau))\Big];\ \tau \leq T, \tau \text{ is a stopping time}\right\}.$$

(v) On the other hand, if the option holder plans to maximize her gain on the option, the only "strategy" available to her is to choose judiciously an optimal stopping time (since she cannot look into the future) to exercise the option. (For example, exercise the option when S_t has reached a low enough level $L < K$, if the option is American put). Therefore, one would not be willing to pay V_0 for the American option if she could not find an exercise time τ^* so that $V_0 = V_0^{\tau^*}$. That is

$$V_0 = E\Big[e^{-r\tau^*} g(S(\tau^*))\Big].$$

(vi) It follows that

$$V_0 = \max\left\{E\Big[e^{-r\tau} g(S(\tau))\Big];\ \tau \leq T, \tau \text{ is a stopping time}\right\}. \tag{6}$$

We shall take this formula as the *definition* of the value of the American option.

(vi) Actually, we have assumed in our argument of part (v) and (vi) that the maximum and the optimal exercise time τ^* exist. If they do not, we must use the supremum, and so the proper definition is

$$V(0) = \sup\left\{\left[e^{-r\tau}g(S(\tau))\right];\, \tau \leq T, \tau \text{ is a stopping time}\right\}. \tag{7}$$

(vii) The definition for the option value at time t is similar.

$$V(t) = \sup\left\{E\left[e^{-r(\tau-t)}g(S(\tau))\middle|S(t)\right];\, \tau \text{ satisfies (4) and (5).}\right\}. \tag{8}$$

(viii) Note that at a theoretical level, definition (8) tells us how to price and also when to exercise the American option. We say this is theoretical since at this point we do not know what V_t is. But assuming that we know V_t then the following principles apply.
At each time t, if $V_t > g(S_t)$ then continue to hold the option (since this implies there is a future exercise time $\tau > t$ that will give one a higher expected pay off than the immediate realization of $g(S_t)$).
If $V_t = g(S_t)$ then exercise the option since this is the best possible value one can realize at time t (among all other possible strategies that make one wait until a future time $\tau > t$ to exercise).

(ix) Define

$$v(t,x) = \sup\left\{E\left[e^{-r(\tau-t)}g(S(\tau)) \;\middle|\; S(t) = x\right];\;\; \tau \text{ satisfies (4) and (5)}\right\} \tag{9}$$

From the perspective of time t this is the best, discounted, expected payoff the option can yield. Then by the Markov property of $S(t)$:

$$V(t) = v(t, S(t)) \tag{10}$$

The function $v(t,x)$ is called the the *value function* of the option pricing problem. A stopping time τ^* for which

$$v(t,x) = E\left[e^{-r(\tau^*-t)}g(S(\tau^*)) \;\middle|\; S(t) = x\right]$$

is called an *optimal exercise*, or *optimal stopping time*, for valuing the option starting at t.

2.3 Our goal

Item (ix) above characterizes the value of the American option ($v(t, S_t)$) and when to exercise (τ^*). Our goal for this note is to find an equation (a PDE) that $v(t, x)$ sastisfies and characterize τ^* (give a rule for τ^*). This will be accomplised in the sections on the perpetual put and American option with finite time of observation. Before that, we still need to discuss more about some properties of V_t.

3 The super-martingale property of V_t

3.1 Introduction

Definition 3.1. *Let X_t be a stochastic process and $\mathcal{F}(t)$ a filtration for X. We say that X_t is a* **super-martingale** *with respect to $\mathcal{F}(t)$ if for $s \leq t$*

$$E(X_t|\mathcal{F}_s) \leq X_s.$$

We say that X_t is a **sub-martingale** *with respect to $\mathcal{F}(t)$ if for $s \leq t$*

$$E(X_t|\mathcal{F}_s) \geq X_s.$$

Example: Let W_t be a Brownian motion and let

$$\begin{aligned} X_t^1 &= t + W_t; \\ X_t^2 &= -t + W_t. \end{aligned}$$

Then X_t^1 is a sub-martingale and X_t^2 is a super-martingale w.r.t. $\mathcal{F}_t^W$.

Remark 3.2. *Note that the definition of super and sub martingale allows for equality. Thus X_t is a martingale if and only if it is both a super-martingale and a sub-martingale.*

Remark 3.3. *Note that a probability measure is also implicitly involved in the definition of super-martingale and sub-martingale. In this note, we will always consider the context under the risk neutral measure (so that $e^{-rt}S_t$ is a martingale under our default measure). However, observe from the above example, that a super-martingale may no longer be so under a change of measure (say we change the drift of the Brownian motion to 2t, for example, then X^2 will be come a sub-martingale under the new measure).*

3.2 Main results

We need to introduce some new notation. For a fixed $t \in [0, T]$, let $\mathcal{T}_{[t,T]}$ be the set of stopping times that satisfy (4) and (5). That is, $\mathcal{T}_{[t,T]}$ is a set of stopping times taking values in $[t, T]$ and adapted to the filtration generated by S_u for time $t \leq u \leq T$. Then observe that for $s \leq t$

$$\mathcal{T}_{[t,T]} \subseteq \mathcal{T}_{[s,T]}.$$

As time goes on (when t increases), the set of strategies available for the option holder (the stopping times) decreases. This intuitively suggests that on average (that is when taking conditional expectation), V_t should be decreasing (because if $A \subseteq B$ then $\sup B \leq \sup B$). That is V_t is a super-martingale.
The rigorous proof for this result is not easy. However, if we accept the fact that for any time t, there will exists $\tau^* \in \mathcal{T}_{[t,T]}$ such that

$$V_t = E\Big[e^{-r(\tau^*-t)} g(S(\tau^*)) \Big| \mathcal{F}_t\Big],$$

then we can show that V_t is a super-martingale. (Later on we will show by an independent result that such a τ^* always exists. This discussion is a motivation for the rigorous results that will follow).

Theorem 3.4. *Assuming the optimal stopping time exists for each V_t, then the discounted American option value $e^{-rt}V_t$ is a super-martingale w.r.t $\mathcal{F}_t^S$.*

Proof. Let $s \leq t$. Let $\tau^* \in \mathcal{T}_{[t,T]}$ such that

$$V_t = E\Big[e^{-r(\tau^*-t)} g(S(\tau^*)) \Big| \mathcal{F}_t\Big],$$

Then

$$\begin{aligned} E(e^{-rt}V_t|\mathcal{F}_s^S) &= E\Big\{E(e^{-r\tau^*} g(S_{\tau^*})|\mathcal{F}_t)\Big|\mathcal{F}_s^S\Big\} \\ &\leq \sup_{\tau\in\mathcal{T}_{[s,T]}} E\Big\{E(e^{-r\tau} g(S_\tau)|\mathcal{F}_t)\Big|\mathcal{F}_s^S\Big\} \\ &= \sup_{\tau\in\mathcal{T}_{[s,T]}} E(e^{-r\tau} g(S_\tau)\Big|\mathcal{F}_s^S) \\ &= e^{-rs}V_s, \end{aligned}$$

where the inequality follows from the fact that if $\tau^* \in \mathcal{T}_{[t,T]}$ then $\tau^* \in \mathcal{T}_{[s,T]}$.

Since $V_t \geq g(S_t) \forall t$, we say $e^{-rt}V_t$ is a super-martingale dominating $e^{-rt}g(S_t)$. The crucial fact about $e^{-rt}V_t$ is that it is also *the smallest super-martingale dominating* $e^{-rt}g(S_t)$ in the following sense:

Theorem 3.5. *Let X_t be a $\mathcal{F}_t^S$ super-martingale dominating $e^{-rt}g(S_t)$, that is $X_t \in \mathcal{F}_t^S$ a super-martingale and $X_t \geq e^{-rt}g(S_t)$ for all t. Then for all t, with probability 1:*

$$X_t \geq e^{-rt}V_t.$$

Proof. Let X_t be a $\mathcal{F}_t^S$ super-martingale dominating $e^{-rt}g(S_t)$. Then it also follows that $X_\tau \geq e^{-r\tau}g(S_\tau)$ for all stopping time τ. We have,

$$\begin{aligned} e^{-rt}V_t &= \sup_{\tau \in \mathcal{T}_{[t,T]}} E(e^{-r\tau}g(S_\tau)\Big|\mathcal{F}_t) \\ &\leq \sup_{\tau \in \mathcal{T}_{[t,T]}} E(X_\tau\Big|\mathcal{F}_t) \\ &\leq \sup_{\tau \in \mathcal{T}_{[t,T]}} X_t. \\ &= X_t \end{aligned}$$

We have used the optinal sampling theorem in the inequality

$$E(X_\tau\Big|\mathcal{F}_t) \leq X_t,$$

which basically says the super-martingale property of X_t also applies when we use a bounded stopping time $\tau \geq t$.

3.3 Implication

The result that V_t is the smallest super-martingale dominating $g(S_t)$ is the key observation to solve the optimal stopping problem. To illustrate the use of the idea, we'll consider the simple senario where everything is deterministic. Then V_t being a super-martingale is just being a decreasing (i.e. non-increasing) function in t.

Example 1: Let $T = 1$ and $g(t) = t$ on $[0, 1]$. What is the smallest decreasing function dominating $g(t)$ on $[0, 1]$? You'll see quickly that it's the constant function $V_t = 1$ on $[0, 1]$.

Example 2: Let $T = 2$ and $g(t) = t$ on $[0, 1]$ and $2 - t$ on $[1, 2]$. What is the smallest decreasing function dominating $g(t)$ on $[0, 1]$? You'll see quickly that it's the function $V_t = 1$ on $[0, 1]$ and $V_t = 2 - t$ on $[1, 2]$.

So what's the observation here? In both cases, V_t is **always a constant** up to the first time it hits $g(t)$. Moreover, we can find out the value of V_0 from the value of $V(t^*)$, where t^* is the point that V_t meets $g(t)$.
Going back our stochastic setting. Then it is not hard to guess that being a constant in the deterministic setting "corresponds" to being a martingale in the stochastic setting. That is V_t should be a martingale up to the first time it hits $g(S_t)$. Moreover, the value of V_0 can be deduced by taking expectation of $E(V_{\tau^*})$, where τ^* is the first time V_t hits $g(S_t)$. All of this argument is intuitive of course, but it serves to illustrate the idea and will be made rigorous in the next section.

4 The American perpetual put

4.1 The value function

The perpetual put is defined by the payoff function $(K - x)^+$ and the expiration date $T = \infty$. In this case, exercise at time ∞ represents no exercise. To include no exercise in the simplest manner, we make the convention that

$$e^{-r\infty}(K - S(\infty))^+ = 0$$

This convention is consistent with taking a limit as $t \to \infty$; indeed

$$\lim_{t\to\infty} e^{-rt}(K - S(t))^+ = 0, \tag{11}$$

because $0 \leq (K - x)^+ \leq K$ for all $x \geq 0$.
For the rest of this discussion, S is the Black-Scholes price with risk free interest rate r and volatility σ. As usual P denotes the risk-neutral measure.

Define $v(t, x)$ as in (9), except now $T = \infty$ and τ is allowed to take the value ∞. Denote $v(0, x)$ by $v(x)$: thus,

$$v(x) = \sup\left\{E\left[e^{-r\tau}(K - S(\tau))^+ \;\middle|\; S(0) = x\right];\, \tau \text{ satisfies (4) and (5) with } T = \infty.\right\}$$

Because the expiration date is infinite and the volatility and risk free rate are constant in time, all times should look the same for the perpetual put. That is, if today $S(0) = 10$ and the price of the perpetual put is 5, it should be 5 whenever the price is 10 in the future.
More precisely, observe that,

(i) Under P, $S(u) = S(0)\exp\{\sigma W(u) + (r - \frac{\sigma^2}{2})u\}$ where $\{W(u); u \geq 0\}$ is a Brownian motion independent of $S(0)$, and

(ii) $S(t+u) = S(t)\exp\{\sigma(W(t+u) - W(t)) + (r - \frac{\sigma^2}{2})u\}$, where, for fixed t, $\{W(t+u) - W(t);\ u \geq 0\}$ is a Brownian motion independent of $S(t)$.

Therefore, the *process* $\{S(t+u);\ u \geq 0\}$ conditioned on $S(t) = x$ is **identical in distribution** to the process $\{S(u);\ u \geq 0\}$ conditioned $S(0) = x$.

Since the value $v(t,x)$ *depends only on the distribution of the price process forward in time*, conditioned on $S(t) = x$, $v(t,x)$ is independent of t. This proves:

Lemma 1. *For the perpetual put, $v(t,x) = v(x)$ for all $t \geq 0$.*

4.2 Martingale characterization

The next theorem is a rigorous statement of the intuitively derived martingale characterization of the price introduced in section 3 above.

Theorem 1. *(Martingale sufficient conditions for the value function.) Assume $u(x)$, $x \geq 0$ satisfies the following conditions:*

(a) $u(x) \geq (K-x)^+$ *for all* $x \geq 0$*;*

(b) *u is bounded—there is a constant $M < \infty$ such that $u(x) \leq M$ for all $x \geq 0$;*

(c) $e^{-rt}u(S(t))$ *is a* **supermartingale** *given any initial condition $S(0) = x$;*

(d) *if τ^* is the first time that $u(S(t)) = (K - S(t))^+$, then $e^{-r(\tau^* \wedge t)}u(S(\tau^* \wedge t))$ is a* **martingale** *given any initial condition $S(0) = x$;*

Then $u(x) = v(x)$, the value function for the perpetual put, and $\tau*$ is the optimal exercise time.

Note: We will be able to find such a u, so, in fact, conditions (a)—(d) uniquely characterize v.

Proof. To prove equality of u and v we will first prove $u(x) \geq v(x)$ for all x, and then prove $u(x) \leq v(x)$.

(i) $u(x) \geq v(x)$:

It suffices to show that

$$u(x) \geq E\left[e^{-r\tau}(K - S(\tau))^+ \;\Big|\; S(0) = x\right] \quad \text{for any stopping time } \tau. \tag{12}$$

Indeed, (12) would imply that

$$u(x) \geq \sup\left\{E\left[e^{-r\tau}(K - S(\tau))^+ \;\Big|\; S(0) = x\right];\, \tau \text{ satisfies (4) and (5) with } T = \infty\right\} = v(x).$$

So let τ be any stopping time and assume $S(0) = x$. We have,

$$\begin{aligned} u(x) &= e^{-r(\tau\wedge t)}u(S(\tau \wedge t))\Big|_{t=0} \geq E\left[e^{-r(\tau\wedge t)}u(S(\tau \wedge t)) \;\Big|\; S(0)=x\right] \\ &\geq E\left[e^{-r(\tau\wedge t)}\Big(K - S(\tau \wedge t)\Big)^+ \;\Big|\; S(0)=x\right], \end{aligned}$$

where the first inequality follows from assumption (c), $e^{-rt}u(S(t))$ is a supermartingale, and by the optional stopping theorem for supermartingales, $e^{-r(\tau\wedge t)}u(S(\tau \wedge t))$ is also a supermartingale and supermartingales decreasing in expectation. The second inequality follows from assumption (a): $u(S(\tau \wedge t)) \geq (K - S(\tau \wedge t))^+$.
Because $(K - S(\tau \wedge t))^+$ is bounded above by K and $\lim_{t\to\infty} e^{-rt}(K - S(t))^+ = 0$, limits and expectation can be interchanged in the above equation; this is a consequence of the dominated convergence theorem and the convention that the value of the option at $t = \infty$ is 0. Therefore,

$$u(x) \geq E\left[\lim_{t\to\infty} e^{-r(\tau\wedge t)}\Big(K - S(\tau\wedge t\Big)^+ \;\Big|\; S(0)=x\right] = E\left[e^{-rt}\Big(K - S(\tau)\Big)^+ \;\Big|\; S(0)=x\right].$$

This proves (12) and so finishes the proof that $u(x) \geq v(x)$.

(ii) $u(x) \leq v(x)$:

We will show

$$u(x) = E\left[e^{-r\tau^*}(K - S(\tau^*))^+ \;\Big|\; S(0) = x\right] \tag{13}$$

where $\tau*$ is defined as in part (d) of the Theorem statement. The value function $v(x)$ is the maximum discounted expected payoff and so it is certainly greater than or equal than the right-hand side of (13). Hence $u(x) \leq v(x)$ follows from (13).

To prove 13 we need only repeat the previous calculation with $\tau*$ replacing τ. But now since $e^{-r(\tau*\wedge t)}u(S(\tau^* \wedge t))$ is a martingale, all inequalities are replaced by equalities and the result is (13). This completes the proof. $\diamond$

4.3 Equations for the value function

Theorem 1 assumed only that the price process was a path continuous Markov process. The following theorem provides analytic conditions on a function u in order that it satisfy the martingale conditions of Theorem 1, in the special case that S follows that Black-Scholes price model. Recall that under the risk-neutral measure,

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dW(t).$$

The theorem provides an effective way of finding the value function or at least obtaining differential equations for the value function. It can easily be extended to other price models.

Theorem 2. *(Hamilton-Jacobi-Bellman equations for the value function.) Assume $u(x)$, $x \geq 0$, satisfies the following conditions:*

(a') $\quad u(x) \geq (K-x)^+$ *for all* $x \geq 0$*;*

(b') $\quad u$ *is bounded—there is a constant* $M < \infty$ *such that* $u(x) \leq M$ *for all* $x \geq 0$*;*

(c') $\quad u$ *and* u' *are continuous and* u'' *is continuous except possibly at a finite number of points, where it has jump discontinuities, and it satisfies*

$$ru(x) - rxu'(x) - \frac{1}{2}\sigma^2 x^2 u''(x) \geq 0; \tag{14}$$

(d') $\quad$ *on the set where* $u(x) > (K-x)^+$ *(the continuation set),*

$$ru(x) - rxu'(x) - \frac{1}{2}\sigma^2 x^2 u''(x) = 0. \tag{15}$$

Then $u(x) = v(x)$, the value function for the perpetual put, and the optimal exercise time is the first time τ^* that $S(t)$ hits the set $\{x;\, v(x) = (K-x)^+\}$.

Condition (a') and the two equations in (c') and (d') are all contained in the following equation:

$$\min\{u(x) - (K-x)^+,\, ru(x) - rxu'(x) - \frac{1}{2}\sigma^2 x^2 u''(x)\} = 0. \tag{16}$$

Remark 4.1. *The condition that u'' is continuous except possibly at a finite number of points may be strange at the beginning, because we're used to assuming that u has continuous second derivatives. This is because for this type of PDE, making the assumption that $C^{1,2}$ is no longer appropriate. To get a clue on why, note that this PDE is* **nonlinear** *in u (equation (16)) while the PDEs we considered so far (in chapter 7 and 11) is* **linear** *in u. This should lead you to expect that the type of regularity property we are expecting for this type of PDE is weaker than the type of regularity we expect for the PDE we encountered in chapter 7. Hence assumption c'.*

Remark 4.2. *The Ito's formula we have learned so far assumes that $u \in C^{1,2}$. We want to apply Ito's formula to the function u in the above theorem. So clearly the question is can we still do so? The answer is essentially yes, via an extension of Ito's rule. We will present it at the end of this section, to preserve the flow of discussion.*

Proof: We need to check that the conditions (a), (b), (c), and (d) of 1 are satisfied by u. Conditions (a) and (b) are automatic from (a') and (b'). For conditions (c) and (d), we use Itô's rule and (c') and (d').

(i) $e^{-rt}u(S(t))$ is a *supermartingale* given any initial condition $S(0) = x$:

Since u'' is assumed continuous except at a finite number of points where it is undefined and has a jump discontinuity, Itô's rule applies and

$$\begin{aligned} e^{-rt}u(S(t)) &= u(S(0)) + \int_0^t e^{-rs}\{-ru(S(s)) + rS(s)u'(S(s)) + \frac{1}{2}\sigma^2 S^2(s)u''(S(s))\}\, ds \\ &\quad + \int_0^t e^{-rs}u'(S(s))\, dW(s). \end{aligned} \tag{17}$$

Let $t_1 < t_2$. The stochastic integral is a martingale so

$$E\left[\int_0^{t_2} e^{-rs}u'(S(s))\, dW(s) \;\Big|\; \mathcal{F}(t_1)\right] = \int_0^{t_1} e^{-rs}u'(S(s))\, dW(s)$$

On the other hand, condition (c') implies that the integrand of the ds integral is always non-positive. Hence,

$$\begin{aligned}
E\Bigg[\int_0^{t_2} & e^{-rs}\{-ru(S(s))+rS(s)u'(S(s))+\frac{1}{2}\sigma^2 S^2(s)u''(S(s))\}\,ds \;\Big|\; \mathcal{F}(t_1)\Bigg] \\
= & \int_0^{t_1} e^{-rs}\{-ru(S(s))+rS(s)u'(S(s))+\frac{1}{2}\sigma^2 S^2(s)u''(S(s))\}\,ds \\
& \quad + E\Bigg[\int_{t_1}^{t_2} e^{-rs}\{-ru(S(s))+rS(s)u'(S(s))+\frac{1}{2}\sigma^2 S^2(s)u''(S(s))\}\,ds \;\Big|\; \mathcal{F}(t_1)\Bigg] \\
\leq & \int_0^{t_1} e^{-rs}\{-ru(S(s))+rS(s)u'(S(s))+\frac{1}{2}\sigma^2 S^2(s)u''(S(s))\}\,ds
\end{aligned}$$

Putting these calculations together,

$$\begin{aligned}
E & \Big[e^{-rt_2}u(S(t_2)) \;\Big|\; \mathcal{F}(t_1)\Big] \\
& \leq u(0) + \int_0^{t_1} e^{-rs}\{-ru(S(s))+rS(s)u'(S(s))+\frac{1}{2}\sigma^2 S^2(s)u''(S(s))\}\,ds \\
& \qquad + \int_0^{t_1} e^{-rs}u'(S(s))\,dW(s) \\
& = e^{-rt_1}u(S(t_1))
\end{aligned}$$

This proves that $e^{-rt}u(S(t))$ is a supermartingale.

(ii) $e^{-r(t\wedge\tau^*)}u(S_{\wedge\tau^*})$ is a *martingale* given any initial condition $S(0)=x$:

Repeat the above calculation but only up to the stopping time τ^*, that is the first time $S(t)$ hits the set $\{x;\, v(x)=(K-x)^+\}$. Then for $t<\tau^*, u(S_t)>(K-S_t)^+$ by definition of τ^* and (a').

So by (d'), for $s<\tau^*$, $-ru(S(s))+rS(s)u'(s)+(1/2)\sigma^2S^2(s)u''(S(s))=0$. Thus from (17), but carried out up to $\tau*$ only,

$$e^{-r(\tau^*\wedge t)}u(S(\tau^*\wedge t)) = u(0) + \int_0^{\tau^*\wedge t} e^{-rs}u'(S(s))\sigma S(s)\,dW(s),$$

and this is a martingale, thereby proving condition (d) of Theorem 1. $\diamond$

4.4 Finding the value function analytically

(i) Guess for the form of u:

To find the value function, we attempt to solve equation (16), or equivalently (14) and (15). The technique is to guess the form of the solution. For the perpetual put one expects that the exercise (stopping) region will have the form $0 \le x \le L^*$ where $0 < L^* < K$; equivalently the continuation region has the form $L^* < x < \infty$. It is intuitively clear that if it is optimal to continue when the price is at a level ℓ, it should also be optimal to continue at all higher prices, because higher the price, the lower is the payoff. Also, one will never exercise when the price is higher than K. Therefore assume the optimal exercise region is $0 \le x \le L^* < K$. By definition of the optimal exercise region,

$$u(x) = (K - x)^+, \quad 0 \le x \le L^*.$$

According to (16) or (15), on $L^* < x < \infty$, $u(x)$ must solve

$$\frac{1}{2}\sigma^2 x^2 u''(x) + rxu'(x) - ru(x) = 0.$$

This is a homogeneous differential equation of Euler type. Observe the equation, since there is x^2 in front of the 2nd derivative and x in front of the 1st derivative, one can guess the general form of the solution if Ax^p, where A is a constant and p is to be determined. Plug this form in to the equation above, we found the equation for p is

$$\frac{1}{2}\sigma^2 p^2 + (r - \frac{1}{2}\sigma^2)p - r = 0.$$

It is easily checked that $p = -\frac{2r}{\sigma^2}$ and $p = 1$ are the solutions to the above quadratic equation. Thus the general solution of the ODE for $u(x)$ has the form,

$$u(x) = Ax^{-2r/\sigma^2} + Bx,$$

where A and B can be arbitrary constants. The condition that u be bounded requires $B = 0$. Thus,

$$u(x) = \begin{cases} K - x, & \text{if } 0 \le x \le L^*; \\ Ax^{-2r/\sigma^2}, & \text{if } x > L^*. \end{cases}$$

(ii) Impose conditions so that u is "smooth":

For this function u to be continuously differentiable, $(K - x)^+$ and Ax^{-2r/σ^2} and their first derivatives must be equal to each other at L^*. First, by matching the expression for u from the left and right of L^*, we get

$$K - L^* = u(L^*-) = u(L^*+) = A(L^*)^{-2r/\sigma^2}. \tag{18}$$

Second, noting that for $x < L^*$, $u'(x) = -1$ and for $x > L^*$, $u'(x) = -\frac{2r}{\sigma^2} A x^{-2r/\sigma^2 - 1}$,

$$-1 = u'(L^*-) = u(L^*+) = -\frac{2r}{\sigma^2} A (L^*)^{-2r/\sigma^2 - 1}. \tag{19}$$

These are called the equations of *smooth fit.* They can be solved uniquely for A and L^*:

$$L^* = \frac{2r}{2r + \sigma^2} K \qquad \text{and} \qquad A = \frac{\sigma^2}{2r} (L^*)^{+2r/\sigma^2 + 1}.$$

Notice that $L^* < K$ as desired.

(iii) Verify that the function u we came up with is the actual solution:
To finish the proof, we must verify that with L^* and A so defined, equation (14) is true on $0 < x < L^*$, that is

$$ru(x) - rxu'(x) - \frac{1}{2}\sigma^2 x^2 u''(x) \geq 0;$$

on $0 < x < L^*$ **and** $u(x) \geq (K - x)^+$ for $L^* < x < \infty$.
It is only necessary to verify (14) on $0 < x < L^*$, since it is true by construction on $x > L^*$, that is we found u on $L^* < x < \infty$ by solving

$$\frac{1}{2}\sigma^2 x^2 u''(x) + rxu'(x) - ru(x) = 0.$$

Similarly, by construction, on $0 < x \leq L^*$, $u(x) \geq (K - x)^+$ (since $L^* < K$).

(iii - a) Verifying equation (14) is true on $0 < x < L^*$:

Since $u(x) = K - x$ on $0 < x < L^*$, it follows by direct calculation that that

$$ru(x) - rxu'(x) - \frac{1}{2}\sigma^2 x^2 u''(x) = r(K - x) + rx = rK > 0,$$

which proves (14).

(iii - b) Verifying that $u \geq (K - x)^+$ on $L^* < x$:

Since $A > 0$, it follows that $u(x) > 0$ for all x. By taking two derivatives of Ax^{-2r/σ^2}, one see easily that it is strictly convex up. A and L^* are chosen so that $u'(L^*) = -1$ and $u(L^*) = K - L^*$. Hence, by convexity, $u'(x) > u'(L^*) = -1$ for all $x > L^*$. Thus if we set $g(x) = u(x) - (K - x)$, it follows that $g(L^*) = 0$ and that $g'(x) > 0$ for $x > L^*$. This means that g is increasing on $L^* < x < \infty$ and hence

$g(x) = u(x) - (K - x) > 0$ on $L^* < x < \infty$. Thus we have shown both that $u(x) > 0$ **and** that $u(x) > K - x$ on $L^* < x < \infty$, which means that $u(x) > (K - x)^+$ on $L^* < x < \infty$.

In conclusion, we have identified the value function, and, more importantly, we have found that the optimal time to exercise the perpetual American put is when the price first hits the region $0 < x < L^*$, where $L^* = 2rK/(2r + \sigma^2)$.

4.5 An extension of Itô's rule

Previously in this course, Itô's rule was stated for $f(Y(t))$, where Y is an Itô process and f is twice continuously differentiable. Actually, Itô's rule will still work so long as $f(x)$ and $f'(x)$ are continuous and $f''(x)$ is defined and continuous everywhere except possibley at a finite number of points, where it has jump discontinuities. As an example, consider the function

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0; \\ -x^2, & \text{if } x < 0. \end{cases}$$

For this function, $f'(x) = 2|x|$, and $f''(x) = 2$ if $x > 0$ and $f''(x) = -2$ if $x < 0$. Thus f'' is not defined at $x = 0$, where it has a jump discontinuity. If W is a Brownian motion, then one can show directly that

$$\begin{aligned} f(W(t)) &= f(0) + \int_0^t f'(W(s))\, dW(s) + \frac{1}{2} \int_0^t f''(W(s)) \mathbf{1}_{\{W(s) \neq 0\}}\, ds \\ &= f(0) + \int_0^t f'(W(s))\, dW(s) + \int_0^t \mathbf{1}_{\{W(s)>0\}}\, ds + \int_0^t -\mathbf{1}_{\{W(s)>0\}}\, ds. \end{aligned}$$

The reason this is true is that Brownian motion spends zero total time at $x = 0$, in the sense that

$$\int_0^t \mathbf{1}_{\{W(u)=0\}}\, du = 0, \quad \text{for all } t > 0, \text{ with probability one.}$$

Therefore, in the approximation arguments that lead to Itô's rule, the failure of f'' to exist at $x = 0$ is not "seen."

More generally, if $A = \{a_1, \ldots, a_n\}$ is any finite collection of points and $X(t) = X(0) + \int_0^t \alpha(u)\, du + \int_0^t \beta(u)\, dW(u)$ and $\beta(t) > 0$ for all t, with probability one, then

$$\int_0^t \mathbf{1}_{\{X(u) \in A\}}\, du = 0, \quad \text{for all } t > 0, \text{ with probability one.} \tag{20}$$

and Itô's rule will be valid for any f for which f and f' are continuous and f'' is piecewise continuous with jump discontinuities at $a_1, \dots, a_n$. That is,

$$f(X(t)) = f(0) + \int_0^t f'(X(s))\, dW(s) + \frac{1}{2}\int_0^t f''(X(s))\mathbf{1}_{\{X(s)\notin A\}}\, ds.$$

For simplicity, we will just write the last term as $\int_0^t f''(X(s))\, ds$; this is reasonable considering the fact (20). It is just necessary to understand that the second derivative can fail to exist at the points of A and we really mean $\int_0^t f''(X(s))\mathbf{1}_{\{X(s)\notin A\}}\, ds$.

In a similar way, Itô's rule extends to functions $f(t,x)$ such that $f(t,x)$, $f_t(t,x)$ and $f_x(t,x)$ are continuous and $f_{xx}(t,x)$ is continuous except possibly along a finite number of curves $x = a_i(t)$, where its value can jump.

4.6 Connection with optimal stopping problem

This section can be seen as giving the answer to the problem: Given that $S_0 = x$, finding the value function

$$V_0 = \max\left\{E\left[e^{-r\tau}(K - S_\tau)^+\right];\, \tau \text{ satisfies (4) and (5) with } T = \infty.\right\}$$

and characterize the optimal stopping time τ^* such that

$$V_0 = E\left[e^{-r\tau^*}(K - S_\tau^*)^+\right].$$

Then the answer is as followed. Solve for the value function $u(x)$ as given in Theorem 2. Define the continuation set as

$$\mathcal{C} = \{x; v(x) > (K - x)^+\}$$

and it is so-called because one should continue (not exercise) so long as $S(t) \in \mathcal{C}$. The stopping set is the complement of $\mathcal{C}$:

$$\mathcal{S} = \{x; v(x) = (K - x)^+\}.$$

When $S(t) \in \mathcal{S}$, it is optimal to exercise at t. We also call $\mathcal{S}$ the optimal exercise region.

Then $V_0 = v(S_0) = v(x)$ and the optimal time to exercise τ^* is

$$\tau^* = \min\{u \geq 0;\ (u, S(u)) \in \mathcal{S}\} \wedge T \tag{21}$$

Note also that $e^{-rt}V_t = e^{-rt}v(S_t)$ is a martingale before time τ^* (that is $e^{-rt\wedge\tau^*}V_{t\wedge\tau^*}$ is a martingale) and a super-martingale in general (that is without stopping at τ^*). This follows from the following observation: if

$$dS_t = rS_tdt + \sigma S_tdWt$$

and u satisfies,

$$ru(x) - rxu'(x) - \frac{1}{2}\sigma^2x^2u''(x) = 0,$$

then $u(S_t)$ is a martingale. Moreover, if

$$ru(x) - rxu'(x) - \frac{1}{2}\sigma^2x^2u''(x) \geq 0,$$

then $u(S_t)$ is a super martingale. The proof is by Ito's formula.
Combined this with the result of Theorem 2, that is on $u(x) > (K-x)^+$

$$ru(x) - rxu'(x) - \frac{1}{2}\sigma^2x^2u''(x) = 0,$$

and

$$ru(x) - rxu'(x) - \frac{1}{2}\sigma^2x^2u''(x) \geq 0,$$

it is easy to see that $u(S_t)$ is a martingale up to the first time $u(S_t)$ hits $(K-S_t)^+$ and generally a super-martingale.

5 American options with finite time of expiration

The analysis that we carried out can be generalized to American options with finite expiration.
As usual, assume the price process of the underlying is Black-Scholes with risk-free rate r and volatility σ^2. Suppose the option payoff is $g(x)$, and assume g is bounded. This assumption is made so that the results we state are rigorously true; extensions are possible that drop the boundedness condition, but then extra conditions are needed.
Let T be the time of expiration and let $v(t,x)$, $t \leq T$, be the value function as defined above in equation (9).
Theorems 1 and 2 have easy generalizations that require little more than inserting extra dependence on t.

5.1 Martingale characterization

Theorem 3. *(Martingale sufficient conditions for the value function.) Assume $u(t,x)$, $x \geq 0$ satisfies the following conditions:*

(a) $u(t,x) \geq g(x)$ *for all* $x \geq 0$, $0 \leq t \leq T$;

(b) u *is bounded—there is a constant* $M < \infty$ *such that* $u(t,x) \leq M$ *for all* $x \geq 0$, $0 \leq t \leq T$;

(c) $e^{-rt}u(t,S(t))$ *is a supermartingale given any initial condition* $S(0) = x$;

(d) *if* τ^* *is the first time that* $u(t,S(t)) = g(S(t))$, *then* $e^{-r(\tau^* \wedge t)}u(\tau^* \wedge t, S_{\tau^* \wedge t})$ *is a martingale given any initial condition* $S(0) = x$;

Then $u(t,x) = v(t,x)$, *the value function for the perpetual put, and* $\tau*$ *is the optimal exercise time.*

5.2 Equation for the value function

Theorem 4. *(Hamilton-Jacobi-Bellman equations for the value function.) Assume $u(t,x)$, $x \geq 0$ satisfies the following conditions:*

(a') $u(t,x) \geq (K-x)^+$ *for all* $x \geq 0$, $0 \leq t \leq T$;

(b') $u(t,x)$ *is bounded on the set* $x \geq 0$, $0 \leq t \leq T$;

(c') $u(t,x)$ *and* $u_t(t,x)$ *and* $u_x(t,x)$ *are continuous and* $u_{xx}(t,x)$ *exists and is continuous except possibly along a finite number of curves* $x = a_i(t)$, *where it has jump discontinuities, and* u *satisfies*

$$ru(t,x) - u_t(t,x) - rxu_x(t,x) - \frac{1}{2}\sigma^2 x^2 u_{xx}(t,x) \geq 0;$$

(d') *on the set where* $u(t,x) > g(x)$ *(the continuation set)*

$$ru(t,x) - u_t(t,x) - rxu_x(t,x) - \frac{1}{2}\sigma^2 x^2 u_{xx}(t,x) = 0.$$

Then $u(t,x) = v(t,x)$, *the value function for the perpetual put, and the optimal exercise time is the first time* τ^* *that* $S(t)$ *hits the set* $\{x;\, v(t,x) = g(x)\}$.

The proofs of these statements mimic closely the proofs written out above for the perpetual put.

5.3 Characterization of the optimal exercise region

For the American put, $g(x) = (K - x)^+$. In this case, explicit formulae for the value function and for the exercise region are not known when T is finite. However, the methodology of Theorems 3 and 4, which can be generalized greatly, is very important. The equations presented in parts (c') and (d') of Theorem 4 can be solved numerically to get prices.

Even though explicit solutions for the value function of the American put are not known, the following can be proved.

The optimal exercise region has the form

$$\{(t, x);\, 0 \le x \le L^*(T - t), \text{for} \;\; x \ge 0,\, 0 \le t \le T\},$$

where $L^*(u)$, $u \ge 0$, is a function defined independently of T satisfying,

(i) $L^*(0) = K$;

(ii) $L^*(u)$ is decreasing as u increases;

(iii) $L^*(u) > \frac{2r}{2r+\sigma^2}K$ for all $u \ge 0$.

All these conditions make good intuitive sense. The parameter u in $L^*(u)$ represents time until expiration. Thus, $L^*(0)$ is the boundary of the exercise region when $t = T$, and at this time one exercises only if the the price is less than K. As u increases, that is, as we go back further and further in time from expiration, the value of the option at all price levels should either increase or stay the same, because more time means a greater range of exercise opportunities. But if the price increases the optimal exercise region shrinks. Finally, as $u \to +\infty$, $L^*(u)$ should decrease to the optimal exercise boundary of the perpetual put, which was shown above to be $\frac{2r}{2r+\sigma^2}K$.